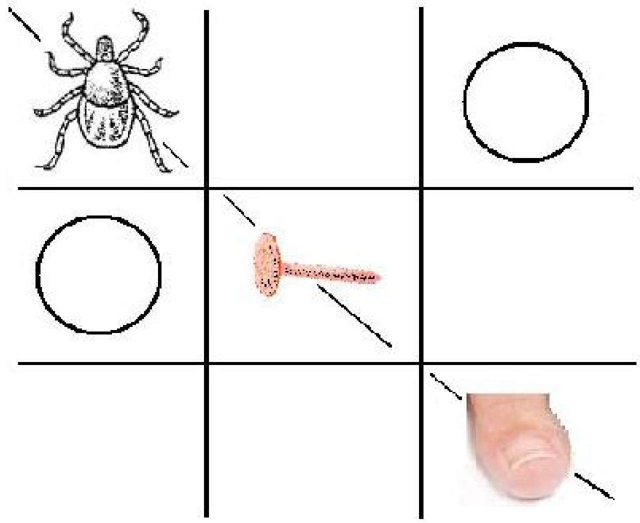
Tic-Tac-Toe



with
Eeny, Meeny, Miny, Moe

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Abstract. 1067-P1-2306 Dennis P. Walsh* (dwalsh@mtsu.edu), Box X070, Middle Tennessee State University, 1301 East Main Street, Murfreesboro, TN 37132.

Tic-Tac-Toe with Eeny, Meeny, Miny, Moe. By allowing various forms of randomness into the game of tic-tac-toe, one can escape the ubiquitous ties that occur when two smart people play the game. Using a simple random device, such as a coin or a die, a person can play tic-tac-toe against a phantom random player. We explore versions of the game that include random play versus random play, random play versus smart play, and smart play versus random play. We will derive the win, lose, and tie probabilities for several variations showing, for example, that the probability of player X winning a random-play-versus-random-play game (under uniform randomness) is approximately .585. These various random games lend themselves nicely to explorations by students, and instructors can use the games to introduce or reinforce basic counting principles. (Received September 22, 2010)

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Famous Opening Moves in Tic-Tac-Toe

1. The Tictackamov Defense

O		
	X	

2. The Tictackamov Variation

		O
	X	

3. The Kid's Gambit

X	O	

4. The Cheating Kid's Gambit

X		
X	O	
X		

Problem 1. **Count the number of final tic-tac-toe grid configurations that are possible when player X (playing first) wins and where board orientation matters.**

Solution. **The number of winning configurations for player X**

Player X Number of Moves	3	4	5	Total
Win Pattern				
H only	45	162	12	219
V only	45	162	12	219
D only	30	120	16	166
HV			9	9
HD			6	6
VD			6	6
DD			1	1
	120	444	62	626

Example. **Player X wins in 4 moves with a horizontal win.**
A typical configuration

X	X	X
	O	O
X	O	

Step	Number of ways
1. Choose row for 3 X's	$\binom{3}{1}$
2. Choose row for the 2 O's	$\binom{2}{1}$
3. Choose cells the 2 O's in step 2	$\binom{3}{1}$
4. Choose cell for 3rd O	$\binom{3}{1}$
5. Choose cell for the 4th X	$\binom{3}{1}$

Problem 2. **Calculate the probability that player X wins a game in which both players play randomly.**

Solution. **We make use of the solution of Problem 1 to calculate $P(W_3)$, $P(W_4)$, and $P(W_5)$ where W_i denotes the event that player X wins on its i-th move. We first note that**

$$P(W_i) = P\left\{\begin{array}{l}\textbf{end-of-game configuration is correct and}\\ i\textbf{-th move by } X \textbf{ wins the game}\end{array}\right\}$$

$$= \frac{\textbf{\# of winning configurations for } W_i}{\textbf{\# of all possible configurations}} \times P(i\textbf{-th X wins game}).$$

Thus

$$P(W_3) = \frac{\textbf{\# of win-in-3 configurations}}{\textbf{number of total 5-move configurations}} P(\textbf{third X on winning line})$$

$$= \frac{120}{\binom{9}{3}\binom{6}{2}}(1) = \frac{2}{21} \textbf{ and}$$

$$P(W_4) = \frac{\textbf{\# of win-in-4 configurations}}{\textbf{number of total 7-move configurations}} P(\textbf{fourth X on winning line})$$

$$= \frac{444}{\binom{9}{4}\binom{6}{3}}\left(\frac{3}{4}\right) = \frac{37}{140}.$$

Since $P($fifth X wins the game$)$ depends on the type of win, we have

$$P(W_5) = \frac{\textbf{\# of win-in-5, single-line-win configurations}}{\textbf{number of total 9-move configurations}} P(\textbf{fifth X on winning line})$$

$$+ \frac{\textbf{\# of win-in-5, double-line-win configurations}}{\textbf{number of total 9-move configurations}} P(\textbf{fifth X on both winning lines})$$

$$= \frac{40}{\binom{9}{5}\binom{4}{4}}\left(\frac{3}{5}\right) + \frac{22}{\binom{9}{5}\binom{4}{4}}\left(\frac{1}{5}\right) = \frac{71}{315}.$$

Therefore, $P(\textbf{Player } X \textbf{ wins}) = \frac{2}{21} + \frac{37}{140} + \frac{71}{315} = \frac{737}{1260} \simeq .585.$

Problem 3. **Calculate the probability that player X wins a game in which player X is smart and player O is random.**

Solution. **A smart player X will play the center cell. With probability $\frac{1}{2}$, random player O will play a side-middle cell and will subsequently lose to the smart player X. On the other hand, with probability $\frac{1}{2}$, random player O will play a side-middle cell and have an opportunity to tie with subsequent blocking moves as illustrated below.**

Probability — **Typical configuration**

$\frac{1}{2}$

O_1		
	X_1	

$\frac{1}{6}$

O_1		O_2
	X_1	
X_2		

$\frac{1}{4}$

O_1	X_3	O_2
	X_1	
X_2	O_3	

$\frac{1}{2}$

O_1	X_3	O_2
X_4	X_1	O_4
X_2	O_3	

Tie

O_1	X_3	O_2
X_4	X_1	O_4
X_2	O_3	X_5

Therefore $P(\textbf{Player } X \textbf{ wins}) = 1 - P(tie) = 1 - \left(\frac{1}{2}\right)\left(\frac{1}{6}\right)\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{95}{96}.$

Problem 4. **Calculate the probability that player O wins a game in which player X is random and player O is smart. Calculate the probability of a tie.**

Solution. **We calculate the probability of player O winning by conditioning on random player X's first move.**

$$P(\textbf{Player } O \textbf{ wins}) = \tfrac{1}{9} P(\textbf{Player } O \textbf{ wins} | X \textbf{ plays center cell})$$

$$+ \ \tfrac{4}{9}\ P(\textbf{Player } O \textbf{ wins} | X \textbf{ plays a corner cell})$$

$$+ \ \tfrac{4}{9}\ P(\textbf{Player } O \textbf{ wins} | X \textbf{ plays a side-middle cell}).$$

$$= \tfrac{1}{9}\left(\tfrac{76}{105}\right) + \ \tfrac{4}{9}\ \left(\tfrac{94}{105}\right) + \ \tfrac{4}{9}\left(\tfrac{19}{21}\right)$$

$$= \tfrac{832}{945} = .8804232804 \approx .88$$

$$P(\textbf{the game is a tie}) = 1 - \tfrac{832}{945} = \tfrac{113}{945} \approx .12$$

Example of one such game. **With probability $\frac{1}{9}$, random player X plays the center cell and smart player O plays a corner:**

O_1		
	X_1	

.

Next, with probability $\frac{1}{7}$, random player X plays the right-column middle cell and player O blocks:

O_1		
O_2	**X_1**	**X_2**

.

With probability $\frac{1}{5}$, random player X plays bottom-row middle cell and subsequently loses when smart player O plays the bottom-row first cell:

O_1		
O_2	**X_1**	**X_2**
O_3	**X_3**	

.

Random Tic Tac Toe Results

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- **Random Player X versus Random Player O**

Event	Probability
X wins in 5	$2/21 \approx .095$
X wins in 7	$37/140 \approx .264$
X wins in 9	$71/315 \approx .225$
X wins	$737/1260 \approx .585$
O wins in 6	$37/420 \approx .088$
O wins in 8	$1/5 = .200$
O wins	$121/420 \approx .288$
Tie	$8/63 \approx .127$
Game ends	1

- **Smart Player X versus Random Player O**

Event	Probability
X wins in 5	$5/6$
X wins in 7	$7/48$
X wins in 9	$1/96$
X wins	$95/96$
Tie	$1/96$

- **Random Player X versus Smart Player O**

Event	Probability
O wins in 6	244/315
O wins in 8	$20/189$
O wins	$832/945$
Tie	$113/945$

- **Board Configurations (Oriented)**

Event	Number of Configurations
X wins in 5	120
O wins in 6	148
X wins in 7	444
O wins in 8	168
X wins in 9	62
Tie	16
Total	958

An Almost Fair Random Game
A coin is flipped prior to each move. If a head occurs, the player plays her mark. If a tail occurs, the player must play her opponent's mark. The game ends as usual – when three-in-a-line occurs or when a tie occurs.

A Fair Random Game
Flip a coin to see who is player X and who is player O. Then follow the “almost fair game ” rules.

A Non-Random Game to Explore
The game is played according to all but one of the traditional rules. The exception occurs when player O plays two O's on her first move. Will a tie always occur when two smart players contend?

A Hybrid Game
The first 6 moves are done traditionally. A die is rolled to determine the cell location for placement of player's X fourth X. The empty cells are numbered left to right, from top to bottom. The die result (mod 3) determines the cell for the fourth X. The final two moves are done traditionally.

Concluding notes . . .

- **These pages are available at**

  http://frank.mtsu.edu/~dwalsh/TICBIG.pdf

- **Detailed calculations for counting all tic-tac-toe board configurations are at**

  http://frank.mtsu.edu/~dwalsh/TIC_FIG.pdf

- **Calculations for counting the permutations associated with every possible result of a random game are at**

  http://frank.mtsu.edu/~dwalsh/TIC_PERM.pdf

- **Play tic tac toe online at**

  http://www.pyzam.com/toys/view/tictacscare

- **Belated acknowledgement. In 1959, T.M. Little of the University of California provided the solution to "Problem E1324 [1) What is the probability that the first player wins? 2) That the second player wins?] in the American Mathematical Monthly, June-July 1958, Vol. 65, No. 6, p. 447. ...(the) solution ... was published on pp. 144-145 of the February 1959 issue." (Butch Malahide), http://sci.tech-archive.net/Archive/sci.math/2005-06/msg05259.html**