Rouletticianː article #6

Runs & Gaps

Any sequence of outcomes will consist of runs and 'gaps'. A run is normally thought of as a *series* of consecutive outcomes of a particular 'class' (say, column 1). However, for our purposes we'll include a 'series' of just one outcome as also being a run. For example, consider a sequence of dozen outcomes (each dozen is denoted by 1, 2, or 3):

$$3 - 22 - 1 - 3 - 2 - 3 - 11 - 2 - 3$$

The runs are separated by hyphens, and there are

1. 2 runs of dozen 1: the first being a run of 1 and the second being a run of 2.
2. 3 runs of dozen 2: the first is a run of 2, the 2nd a run of 1 and the 3rd a run of 1.
3. 4 runs of dozen 3: each being a run of 1.

Similarly, I'll define a gap as the *absence* of any run for a particular class. So a gap is just a run or series of consecutive outcomes in which the outcome of interest does *not* occur. For example, using the same sequence as above, I'll denote the gaps by G. So with respect to dozen 3 the sequence of gaps and runs is:

$$3 - GGG - 3 - G - 3 - GGG - 3$$

There are 3 gaps with respect to dozen 3: the first is a gap of 3, the 2nd a gap of 1, and the 3rd another gap of 3.

Having defined runs and gaps, we can proceed to calculate some statistics concerning them.

Waiting Time

One relevant statistic is the average number of spins or average waiting time required to get a run or gap of a particular length. This number depends on two things: the length of the run, denoted by r , and the probability of success for a *single* outcome in a given class, which I'll denote by p . For example, in the first sequence of dozen outcomes above, we had to wait for 9 spins before a run of 2 for dozen 1, and in the second sequence the waiting time was 4 spins before a gap of length 3 (note that the waiting time is defined as the number of spins which *includes* the gap or run, not the number of spins at which it starts).

This information might be of interest if you're planning to increase your stake on a bet after a win, hoping for a repeat or number of repeats.

Here's the formula you need to use to get the average number of spins before encountering a run of length r :

$$\text{Average waiting time} = \frac{1 - p^r}{(1-p) \cdot p^r}$$

As an example, let's use the above sequences to calculate the waiting times for **(a)** a run of length 2 for a single dozen, and **(b)** a gap of length 3 for two dozens.

(a) The run length, r is 2, and the probability, p is $12/37$. Also, $1 - p = 25/37$. Inserting these values into the formula we get:

$$\text{Waiting time} = \frac{1 - \left(\frac{12}{37}\right)^2}{\left(\frac{25}{37}\right) \times \left(\frac{12}{37}\right)^2} \approx 12.59$$

which we round up to 13 spins.

(b) The run length we require is 3, so $r = 3$. In this case the probability is for two dozens, so $p = 24/37$, and $1 - p = 13/37$. The formula gives us:

$$\text{Waiting time} = \frac{1 - \left(\frac{24}{37}\right)^3}{\left(\frac{13}{37}\right) \times \left(\frac{24}{37}\right)^3} \approx 7.58$$

which rounded up gives us a waiting time of 8 spins.

Of course, we know that the *average* waiting time doesn't give us the full story; how much can the waiting time vary from the average? For example, on average you'll need to wait 15 spins for 3 consecutive reds to come up, but it may happen on the first 3 spins, or you may have to wait for 50 or more spins. We can quantify our uncertainty about this by using the standard deviation, the formula for which is:

$$\text{Standard Deviation of waiting time} = \sqrt{\frac{1}{(q \cdot p^r)^2} - \frac{2 \cdot r + 1}{q \cdot p^r} - \frac{p}{q^2}}$$

where $q = 1 - p$.

This is a pretty hairy formula, but don't panic because the calculator below will do the work for you. The standard deviation enables us to estimate an interval of waiting times, and also tells us how confident we can be that a given run length will occur within that interval. Just enter values for p and r , and click CALCULATE. The calculator will output the average waiting time and an interval within which there is at least a 90% chance that your actual waiting time will fall.

Waiting Time for a Run

Enter the run length r :

Enter the probability (as a decimal) of success on one trial, p :

CALCULATE

The average waiting time in spins is

There is at least a 90% chance that the waiting time will fall between **X** and **Y** spins.

For example, you want to know how many spins, on average, it will take before a run of 4 on an even chance. Enter 4 in the run length box. The probability of a single even-chance outcome is $18/37 = 0.4865$, so enter 0.4865 in the probability box (**note:** always enter the probability as a decimal, not as a fraction). The calculator will give you an average waiting time of 33 spins, and tell you that there is at least a 90% chance that the waiting time will be between 4 and 123 spins.

Playing around with the calculator and putting different values in will quickly reveal a pattern concerning runs and waiting times: the larger the probability, the shorter the waiting time for a given run to appear. You can see this clearly in the table below, which shows probabilities from 0.1 to 0.9 and their corresponding waiting times for a run of 3.

Probabilities & Waiting Times for a Run of 3

Probability (Chance)	Waiting Time
10%	1110
20%	155
30%	51
40%	24
50%	14
60%	9
70%	6
80%	5
90%	4

Waiting Time for Run or Gap

The formulas given above tell you the waiting time and dispersion for runs when considering a *single* outcome, such as red, dozen 1, street 1-3, etc. There is another formula which looks at the sequence as a whole (runs + gaps), and tells you the waiting time for a run *or* a gap of a certain length to occur.

Here it is:

$$\text{Waiting time for run or gap} = \frac{(1 - p^r) \cdot (1 - q^g)}{p^r \cdot q + p \cdot q^g - p^r \cdot q^g}$$

where again, $q = 1 - p$. The indexes of the probabilities are r and g , which denote run and gap respectively.

For example, how many spins would you have to wait, on average, for a run of 5 reds or a run of 5 blacks? Intuitively, you might think that it would take half as long as waiting for a run of one or the other on its own. This is correct. In fact it takes an average of 62 spins (ignoring the zero) to get a run of 5 reds (or blacks), but only 31 spins to get a run of 5 on either.

However, don't go thinking that this reasoning applies to every bet. It just happens to be true in this case because the chance of black is the same as the chance of red (and the result isn't quite accurate because we've ignored the zero).

To clarify, the formula takes a given probability and then subtracts it from 1 to find the 'complementary' probability. So the implied sequence consists of two values: run and gap. There are only 2 'events' possible and their probabilities must add up to 1. For example, you *cannot* use the formula to find the average waiting time of either a run of 2 on dozen 1 or a run of 3 on dozen 2, because you've ignored the zero and dozen 3, therefore the sum of your probabilities ($12/37 + 12/37$) does *not* add up to 1.

In fact, if you wanted to do the above calculation you don't need the 'either or' formula, because if the probabilities are the same, then the average waiting time will be that of the bet with the shortest run — in this case the run of 2 on dozen 1.

An example of using the formula: what is the waiting time for either a run of 3 of a particular dozen, or a gap of 6?

Solution

The probability, p , of a dozen is $12/37$, so the probability of q is $25/37$ (this is the probability of the remaining outcomes hitting, namely the remaining dozens and the zero). The length of the run, r , is 3 and the length of the gap, g , is 6. Now we plug the numbers into the formula:

$$\text{waiting time} = \frac{\left(1 - \left(\frac{12}{37}\right)^3\right) \cdot \left(1 - \left(\frac{25}{37}\right)^6\right)}{\left(\frac{12}{37}\right)^3 \cdot \left(\frac{25}{37}\right) + \left(\frac{12}{37}\right) \cdot \left(\frac{25}{37}\right)^6 - \left(\frac{12}{37}\right)^3 \cdot \left(\frac{25}{37}\right)^6} = 17 \text{ (rounded)}$$

Again I've provided a calculator; just enter values for p , r , and g then click CALCULATE. You might like to check that the inputs in the example just given do result in a waiting time of 17. Note that p goes with r , so put 0.324 into the p box (the decimal equivalent of $12/37$), 3 into the r box, and 6 into the g box. If you put the r and g into the wrong boxes you'll get a different answer.

Waiting Time (Run or Gap)

Enter the probability (as a decimal) of success on one trial, p :

Enter the run length, r :

Enter the gap length, g :

CALCULATE

The average waiting time in spins for a run of **R** or a gap of **G** is

---

Probability of Waiting Time

Finally, it would be useful to know the probability that a run of a given length (or longer) for a certain bet, such as an even chance or dozen, will occur in a given number of trials. For example, the probability that a run of at least 4 reds will occur within 30 spins or the probability that a run of at least 3 for dozen 1 will occur within 20 spins.

Unfortunately there is no tidy formula for calculating this probability, although there is an algorithm which gives a precise answer. The algorithm is implemented in the following calculator.

Probability of Runs

Enter number of trials, n :

Enter run length, r :

Enter probability of success on a single trial (as a decimal):

CALCULATE

Probability of a run of at least **R** in **N** trials is

Try putting the values in for those examples I just mentioned. The probability of a run of at least 4 reds in 30 spins is 60.3%, and the probability of a run of at least 3 of a particular dozen in 20 spins is 36.9%.